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Inversion identities for the self-dual Potts and Ashkin-Teller models

Paul A Pearce
Mathematics Department, University of Melbourne, Parkville, Victoria 3052, Australia

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Abstract. The commuting diagonal-to-diagonal transfer matrices of the self-dual Potts and Ashkin-Teller models on the square lattice are shown to satisfy special functional equations called inversion identities. These identities generalise the known local inversion or unitarity relations for interaction-round-a-face or IRF models satisfying Yang-Baxter or star-triangle equations.

1. Introduction

It has recently been pointed out (Pearce 1987) that the commuting row-to-row transfer matrices of many exactly solvable interaction-round-a-face (IRF) models (Baxter 1982a) satisfy functional equations of a special form called inversion identities. These equations generalise the known local inversion or unitarity relations (Baxter 1982a, c). In this paper I will derive inversion identities for the commuting diagonal-to-diagonal transfer matrices of the self-dual, q-state Potts models (Potts 1952) and the self-dual Ashkin-Teller model (Ashkin and Teller 1943) on the square lattice. These results, together with the results discussed in Pearce (1987), strongly suggest that all IRF models satisfying Yang-Baxter or star-triangle equations will possess inversion identities.

Unlike the inversion relations, which they generalise, the inversion identities can be solved for the complete eigenvalue spectrum of the commuting transfer matrices, thus enabling the exact calculation of free energies, correlation lengths and interfacial tensions as well as many critical exponents and scaling dimensions. This work has been carried out for hard hexagons (Baxter and Pearce 1982), interacting hard squares (Baxter and Pearce 1983) and, most recently, magnetic hard squares (Pearce 1985, Pearce and Kim 1987). The methods of solving the inversion identities for the eigenvalue spectra of the row-to-row transfer matrices developed in this series of papers will be applied to the Potts and Ashkin-Teller models in subsequent papers. It is expected that these equations will ultimately yield a complete analytic solution giving many new exact results and confirming directly many of the critical exponents which are believed to be known exactly from renormalisation group arguments (den Nijs 1979, 1981, Knops 1980, Nienhuis 1984). The inversion identities for the Potts and Ashkin-Teller models are derived together here to point out their similarity and to emphasise the scope of this approach. The inversion identity for the Potts models is derived in § 2. The results for the Ashkin-Teller model are presented in § 3.

2. The self-dual Potts models

The Hamiltonian of the square lattice Potts model is

$$H = -J \sum_{\text{horiz}} \delta(\sigma_i, \sigma_j) - K \sum_{\text{vert}} \delta(\sigma_i, \sigma_j) \tag{2.1}$$

where the sums are over the horizontal and vertical bonds of the lattice, each spin $\sigma_i = 1, 2, \ldots, q$ can be in any of q allowed states and

$$\delta(\sigma, \sigma') = \begin{cases} 1 & \sigma = \sigma' \\ 0 & \sigma \neq \sigma' \end{cases} \tag{2.2}$$

is the usual Kronecker delta. The partition function can be written as

$$Z_N = (\rho_1 \rho_2)^{-2N} \operatorname{Tr}(\boldsymbol{V}\boldsymbol{W})^N \tag{2.3}$$

where $\boldsymbol{V}$ and $\boldsymbol{W}$ are $q^N \times q^N$ diagonal-to-diagonal transfer matrices with elements

$$\begin{aligned} V_{\sigma,\sigma'} &= \prod_{j=1}^{N} W_1(\sigma_j, \sigma'_j) W_2(\sigma_{j+1}, \sigma'_j) \\ W_{\sigma,\sigma'} &= \prod_{j=1}^{N} W_1(\sigma_j, \sigma'_{j+1}) W_2(\sigma_j, \sigma'_j) \end{aligned} \tag{2.4a}$$

as shown in figure 1 with edge weights given by

$$\begin{aligned} W_1(\sigma, \sigma') &= \rho_1 \exp[J\delta(\sigma, \sigma')] \\ W_2(\sigma, \sigma') &= \rho_2 \exp[K\delta(\sigma, \sigma')]. \end{aligned} \tag{2.4b}$$

The normalisation factors ρ_1 and ρ_2 are arbitrary and periodic boundary conditions have been applied to the row configurations σ and σ' so that $\sigma_{N+1} = \sigma_1$ and $\sigma'_{N+1} = \sigma'_1$. The transfer matrices $\boldsymbol{V}$ and $\boldsymbol{W}$ commute with the cyclic shift operator $\boldsymbol{C}$ with elements

$$C_{\sigma,\sigma'} = \delta(\sigma_1, \sigma'_2)\delta(\sigma_2, \sigma'_3) \ldots \delta(\sigma_N, \sigma'_1) \tag{2.5}$$

and are related by

$$\boldsymbol{W} = \boldsymbol{V}\boldsymbol{C}. \tag{2.6}$$

The Potts model cannot be solved in general, but it is exactly solvable (Baxter 1973, 1982a, b) on the self-dual manifold

$$(\mathrm{e}^J - 1)(\mathrm{e}^K - 1) = q. \tag{2.7}$$

For our purposes it is convenient to introduce the parameters

$$\Delta = \sqrt{q} = \begin{cases} 2\cos\lambda & 0 \leqslant q \leqslant 4 \\ 2\cosh\lambda & q > 4 \end{cases} \tag{2.8a}$$

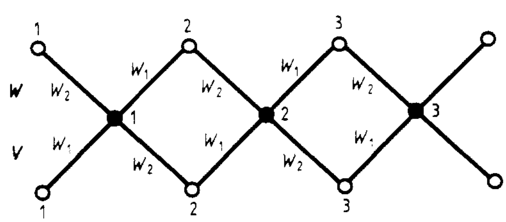

Figure 1. Three successive rows of the square lattice drawn diagonally, showing the diagonal-to-diagonal transfer matrices $\boldsymbol{V}$ and $\boldsymbol{W}$ and the labelling of lattice sites.

and

$$s=\frac{s(u)}{s(\lambda)} \qquad s_-=\frac{s(\lambda-u)}{s(\lambda)} \qquad s_+=\frac{s(\lambda+u)}{s(\lambda)} \qquad 0\leq u\leq\lambda \tag{2.8b}$$

where

$$s(u)=\begin{cases}\sin u & 0\leq q\leq 4\\ \sinh u & q>4.\end{cases} \tag{2.8c}$$

Setting $s_-/s=\Delta^{-1}(e^J-1)=\Delta(e^K-1)^{-1}$, it then follows that the edge weights (2.4*b*) can be written as

$$\begin{aligned}W_1(\sigma,\sigma'|u)&=\Delta^{-1/2}s+\Delta^{1/2}s_-\delta(\sigma,\sigma')=W(\sigma,\sigma'|u)\\ W_2(\sigma,\sigma'|u)&=\Delta^{-1/2}s_-+\Delta^{1/2}s\delta(\sigma,\sigma')=W(\sigma,\sigma'|\lambda-u)\end{aligned} \tag{2.9a}$$

where we have chosen

$$\rho_1=\Delta^{-1/2}s \qquad \rho_2=\Delta^{-1/2}s_- \qquad \rho_1\rho_2=\Delta^{-1}ss_-=(e^{J+K}-1)^{-1}. \tag{2.9b}$$

With this parametrisation, the elements of the diagonal-to-diagonal transfer matrix $\boldsymbol{V}(u)$ are entire functions of u, $\boldsymbol{V}(0)=\boldsymbol{I}$, $\boldsymbol{V}(\lambda)=\boldsymbol{C}^{-1}$ and replacing u with $\lambda-u$ corresponds to interchanging J and K.

With the above parametrisation, the edge weights of the self-dual Potts models possess two important local properties. The first of these is the local inversion (Baxter 1982a, c) or unitarity relations

$$\begin{aligned}&\sum_\tau W_1(\sigma,\tau|u)W_1(\tau,\sigma'|-u)=\Delta s_+s_-\delta(\sigma,\sigma')\\ &W_2(\sigma,\sigma'|u)W_2(\sigma,\sigma'|-u)=\Delta^{-1}s_+s_-.\end{aligned} \tag{2.10}$$

The second local property, shown pictorially in figure 2, is the star-triangle relation (Baxter 1982a)

$$\sum_\tau W(\sigma,\tau|u)W(\sigma',\tau|u')W(\sigma'',\tau|u'')$$
$$=AW(\sigma,\sigma'|\lambda-u'')W(\sigma',\sigma''|\lambda-u)W(\sigma'',\sigma|\lambda-u') \tag{2.11}$$

where A is an arbitrary constant and $u+u'+u''=\lambda$. Since the edge weights satisfy the star-triangle equations on the exact solution manifold (2.7) it follows (Baxter 1982a) that the transfer matrices $\boldsymbol{V}$, $\boldsymbol{W}$ and $\boldsymbol{C}$ all commute with one another.

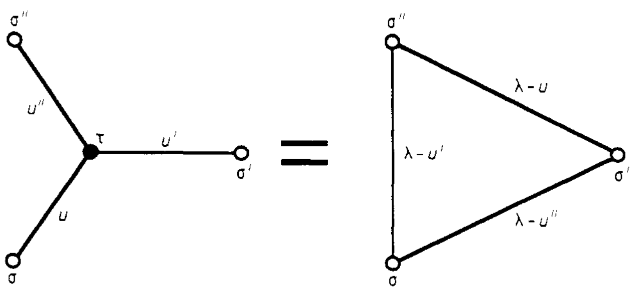

Figure 2. Pictorial representation of the star-triangle equations. The Boltzmann weights of the star (with central spin summed out) and the triangle are equal for all u, u' and u'' satisfying $u+u'+u''=\lambda$.

We will now show that the diagonal-to-diagonal transfer matrix $\boldsymbol{V}(u)$ satisfies the functional equation or inversion identity (Pearce 1987)

$$\boldsymbol{V}(u)\boldsymbol{V}(u+\lambda)\boldsymbol{C} = \phi(\lambda+u)\phi(\lambda-u)\boldsymbol{I} + \phi(u)\boldsymbol{P}(u) \tag{2.12a}$$

where $\boldsymbol{C}$ is the shift operator, $\boldsymbol{I}$ is the $q^N \times q^N$ identity matrix,

$$\phi(u) = \frac{s^{2N}(u)}{s^{2N}(\lambda)} = s^{2N} \tag{2.12b}$$

and $\boldsymbol{P}(u)$ is an auxiliary matrix that commutes with $\boldsymbol{V}(u)$ and whose elements are entire functions of u.

First we observe (see figures 1 and 3) that

$$[\boldsymbol{V}(u)\boldsymbol{V}(u+\lambda)\boldsymbol{C}]_{\sigma,\sigma'} = \prod_{j=1}^{N} \boldsymbol{X}(\sigma_j, \sigma_j') \cdot \boldsymbol{Y}(\sigma_{j+1}, \sigma_{j+1}') \tag{2.13a}$$

where the dot or scalar products involve the q-dimensional vectors

$$\begin{aligned} [\boldsymbol{X}(\sigma, \sigma')]_\tau &= W(\sigma, \tau|u)\, W(\tau, \sigma'|-u) \\ [\boldsymbol{Y}(\sigma, \sigma')]_\tau &= W(\sigma, \tau|\lambda-u)\, W(\tau, \sigma'|\lambda+u). \end{aligned} \tag{2.13b}$$

From (2.9a) it is readily found that

$$\begin{aligned} [\boldsymbol{X}(\sigma, \sigma)]_\tau &= \Delta\delta(\sigma, \tau) - \Delta^{-1}s^2 \\ [\boldsymbol{Y}(\sigma, \sigma)]_\tau &= \Delta^{-1}s_+s_-. \end{aligned} \tag{2.14a}$$

Otherwise, if $\sigma \neq \sigma'$, we have

$$\begin{aligned} [\boldsymbol{X}(\sigma, \sigma')]_\tau &= ss_+\delta(\sigma', \tau) - ss_-\delta(\sigma, \tau) - \Delta^{-1}s^2 \\ [\boldsymbol{Y}(\sigma, \sigma')]_\tau &= ss_+\delta(\sigma, \tau) - ss_-\delta(\sigma', \tau) + \Delta^{-1}s_+s_-. \end{aligned} \tag{2.14b}$$

In particular, using the identities

$$s_+ - s_- = \Delta s \qquad s_+s_- = 1 - s^2 \tag{2.15}$$

we find that, for $\sigma_j \neq \sigma_j'$ and $\sigma_{j+1} \neq \sigma_{j+1}'$,

$$\begin{aligned} \boldsymbol{X}(\sigma_j, \sigma_j) \cdot \boldsymbol{Y}(\sigma_{j+1}, \sigma_{j+1}) &= (s_+s_-)^2 \\ \boldsymbol{X}(\sigma_j, \sigma_j') \cdot \boldsymbol{Y}(\sigma_{j+1}, \sigma_{j+1}) &= 0 \\ \boldsymbol{X}(\sigma_j, \sigma_j') \cdot \boldsymbol{Y}(\sigma_{j+1}, \sigma_{j+1}') &= s^2 \times \text{entire function of } u. \end{aligned} \tag{2.16}$$

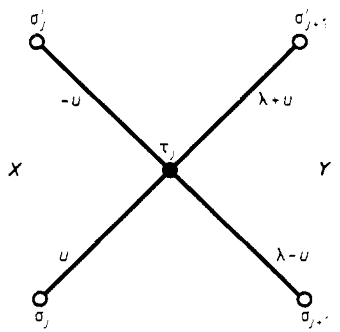

Figure 3. Lattice segments corresponding to the vectors $\boldsymbol{X}(\sigma_j, \sigma_j')$ and $\boldsymbol{Y}(\sigma_{j+1}, \sigma_{j+1}')$. The scalar product is formed by summing out the central spin.

Because of the periodic boundary conditions, it follows that the elements of $V(u)V(u+\lambda)C$ fall into two categories: either $\sigma_j = \sigma'_j$ for all j (these are the diagonal elements), or else $\sigma_j \neq \sigma'_j$ for all j. In all cases we conclude that the elements of $V(u)V(u+\lambda)C$ are of the form given by the inversion identity (2.12) with the elements of $P(u)$ entire functions of u.

3. The self-dual Ashkin–Teller model

The Hamiltonian of the square lattice Ashkin-Teller model that we consider is

$$H = -\sum_{\text{horiz}} [J(a_ia_j + b_ib_j) + J_4a_ia_jb_ib_j] - \sum_{\text{vert}} [K(a_ia_j + b_ib_j) + K_4a_ia_jb_ib_j] \tag{3.1}$$

where the sums are over the horizontal and vertical bonds of the lattice and the Ising spins a_i and b_i take the values ± 1. For notational convenience we assign (Fan 1972a) a compound 4-state spin $\sigma_i = (a_i, b_i)$ to each site i of the lattice. The partition function Z_N can then again be written as in (2.3) with the $4^N \times 4^N$ diagonal-to-diagonal transfer matrices V and W, as shown in figure 1, and defined by (2.4a). For the Ashkin-Teller model the edge weights are given by

$$\begin{aligned} W_1(\sigma, \sigma') &= \rho_1 \exp[J(aa' + bb') + J_4aa'bb'] \\ W_2(\sigma, \sigma') &= \rho_2 \exp[K(aa' + bb') + K_4aa'bb']. \end{aligned} \tag{3.2}$$

The normalisation factors ρ_1 and ρ_2 are at our disposal and periodic boundary conditions are applied to the row configurations σ and σ' so that $a_{N+1} = a_1$, $b_{N+1} = b_1$, $a'_{N+1} = a'_1$ and $b'_{N+1} = b'_1$. The transfer matrices V and $W = VC$ commute with the cyclic shift operator C now defined by (2.5) with

$$\delta(\sigma, \sigma') = \delta(a, a')\delta(b, b'). \tag{3.3}$$

The exact solution manifold of the Ashkin-Teller model is a two-dimensional manifold, in the four-dimensional thermodynamic space spanned by the interactions J, K, J_4 and K_4, given by the constraints

$$\begin{aligned} &\frac{\sinh 2J_4}{\sinh 2J} = \frac{\sinh 2K_4}{\sinh 2K} = \Delta/2 \\ &[\exp(4J_4) - 1][\exp(4K_4) - 1] = \Delta^2. \end{aligned} \tag{3.4}$$

By reversing the spins on one sublattice, if necessary, we can assume that $\Delta \geqslant 0$. As for the Potts model, it is convenient to introduce parameters λ, s, s_-, s_+ given by

$$\Delta = \begin{cases} 2\cos\lambda & 0 \leqslant \Delta \leqslant 2 \\ 2\cosh\lambda & \Delta > 2 \end{cases} \tag{3.5a}$$

and

$$s = \frac{s(u)}{s(\lambda)} \qquad s_- = \frac{s(\lambda - u)}{s(\lambda)} \qquad s_+ = \frac{s(\lambda + u)}{s(\lambda)} \qquad 0 \leqslant u \leqslant \lambda \tag{3.5b}$$

where

$$s(u) = \begin{cases} \sin u & 0 \leqslant \Delta \leqslant 2 \\ \sinh u & \Delta > 2. \end{cases} \tag{3.5c}$$

In this instance we set $s_- = \tanh 2J$ and $s = \tanh 2K$. The edge weights (3.2) can then be written as

$$
\begin{aligned}
W_1(\sigma, \sigma'|u) &= [(1+s) + s_-(aa' + bb') + (1-s)aa'bb']/2\sqrt{2} = W(\sigma, \sigma'|u) \\
W_2(\sigma, \sigma'|u) &= [(1+s_-) + s(aa' + bb') + (1-s_-)aa'bb']/2\sqrt{2} = W(\sigma, \sigma'|\lambda - u)
\end{aligned}
\tag{3.6a}
$$

where we have chosen

$$
\begin{aligned}
\rho_1 &= (s/2)^{1/2}(1-s_-^2)^{1/4} = 2^{-1/2} \exp J_4 \tanh 2K \\
\rho_2 &= (s_-/2)^{1/2}(1-s^2)^{1/4} = 2^{-1/2} \exp K_4 \tanh 2J \\
\rho_1\rho_2 &= 2^{-1}(ss_-)^{1/2}(1-s^2)(1-s_-^2)^{1/4} = 2^{-1} \exp(J_4 + K_4) \tanh 2J \tanh 2K.
\end{aligned}
\tag{3.6b}
$$

Again this parametrisation ensures that the elements of the diagonal-to-diagonal transfer matrix $\boldsymbol{V}(u)$ are entire functions of u. In addition $\boldsymbol{V}(0) = \boldsymbol{I}$, $\boldsymbol{V}(\lambda) = \boldsymbol{C}^{-1}$ and replacing u with $\lambda - u$ corresponds to interchanging J, J_4 with K, K_4 respectively. The local inversion or unitarity relations take the form

$$
\begin{aligned}
&\sum_\tau W_1(\sigma, \tau|u)\, W_1(\tau, \sigma'|-u) = 2s_+s_-\delta(\sigma, \sigma') \\
&W_2(\sigma, \sigma'|u)\, W_2(\sigma, \sigma'|-u) = 2^{-1}s_+s_-.
\end{aligned}
\tag{3.7}
$$

and, moreover, the star-triangle relations (2.11) are satisfied ensuring that the transfer matrices $\boldsymbol{V}$, $\boldsymbol{W}$ and $\boldsymbol{C}$ all commute with one another. For the isotropic model ($J = K$, $J_4 = K_4$) we have $u = \lambda/2$ and

$$
\exp(-2J_4) = \sinh 2J \tag{3.8}
$$

which is the condition for self-duality (Fan 1972b, Baxter 1982a).

We will now show that the diagonal-to-diagonal transfer matrix $\boldsymbol{V}(u)$ of the Ashkin–Teller model satisfies precisely the same inversion identity (2.12) as the Potts model. The elements of $\boldsymbol{V}(u)\boldsymbol{V}(u+\lambda)\boldsymbol{C}$ can be written as simple products as in (2.13) with the four-dimensional vectors $\boldsymbol{X}(\sigma, \sigma') = \boldsymbol{X}(a, b; a', b')$ and $\boldsymbol{Y}(\sigma, \sigma') = \boldsymbol{Y}(a, b; a', b')$ now given by (2.13b) and (3.6a). Explicitly, it is found that

$$
\begin{aligned}
&\boldsymbol{X}(a, b; a, b) = \boldsymbol{U}\begin{bmatrix} s_+s_- \\ b(s_+ + s_-)/2 \\ ab \\ a(s_+ + s_-)/2 \end{bmatrix}
&&\boldsymbol{Y}(a, b; a, b) = \boldsymbol{U}\begin{bmatrix} s_+s_- \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&\boldsymbol{X}(a, b; -a, -b) = s\boldsymbol{U}\begin{bmatrix} 0 \\ -\Delta b/2 \\ abs \\ -\Delta a/2 \end{bmatrix}
&&\boldsymbol{Y}(a, b; -a, -b) = \boldsymbol{U}\begin{bmatrix} 1 \\ bs \\ abs^2 \\ as \end{bmatrix} \\
&\boldsymbol{X}(a, b; -a, b) = s\boldsymbol{U}\begin{bmatrix} 0 \\ \Delta bs/2 \\ -ab \\ -a(s_+ + s_-)/2 \end{bmatrix}
&&\boldsymbol{Y}(a, b; -a, b) = \boldsymbol{U}\begin{bmatrix} (s_+ + s_-)/2 \\ \Delta bs^2/2 \\ \Delta abs/2 \\ as(s_+ + s_-)/2 \end{bmatrix} \\
&\boldsymbol{X}(a, b; a, -b) = s\boldsymbol{U}\begin{bmatrix} 0 \\ -b(s_+ + s_-)/2 \\ -ab \\ \Delta as/2 \end{bmatrix}
&&\boldsymbol{Y}(a, b; a, -b) = \boldsymbol{U}\begin{bmatrix} (s_+ + s_-)/2 \\ bs(s_+ + s_-)/2 \\ \Delta abs/2 \\ \Delta as^2/2 \end{bmatrix}
\end{aligned}
\tag{3.9}
$$

where the column entries are in the order $\tau = (+,+), (-,-), (-,+), (+,-)$ and $\boldsymbol{U}$ is a 4×4 orthogonal matrix given by the direct product

$$\boldsymbol{U} = \tfrac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \tag{3.10}$$

It can be seen immediately from the explicit form of the vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ that, for $\sigma_j \neq \sigma'_j$ and $\sigma_{j+1} \neq \sigma'_{j+1}$, the scalar products again satisfy (2.16). Because of the periodic boundary conditions, it follows that the elements of $\boldsymbol{V}(u)\boldsymbol{V}(u+\lambda)\boldsymbol{C}$ fall into two categories: either $\sigma_j = \sigma'_j$ for all j (these are the diagonal elements), or else $\sigma_j \neq \sigma'_j$ for all j. In all cases we conclude that the elements of $\boldsymbol{V}(u)\boldsymbol{V}(u+\lambda)\boldsymbol{C}$ are of the form given by the inversion identity (2.12) with the elements of $\boldsymbol{P}(u)$ entire functions of u. Thus it has been shown that, with the appropriate parametrisation, the diagonal-to-diagonal transfer matrices of both the self-dual Potts and Ashkin–Teller models satisfy the same inversion identity (2.12).

References

Ashkin J and Teller E 1943 *Phys. Rev.* **64** 178–84

Baxter R J 1973 *J. Phys. C: Solid State Phys.* **6** L445–8

—— 1982a *Exactly Solved Models in Statistical Mechanics* (London: Academic)

—— 1982b *J. Phys. A: Math. Gen.* **15** 3329–40

—— 1982c *J. Stat. Phys.* **28** 1–41

Baxter R J and Pearce P A 1982 *J. Phys. A: Math. Gen.* **15** 897–910

—— 1983 *J. Phys. A: Math. Gen.* **16** 2239–55

den Nijs M P M 1979 *J. Phys. A: Math. Gen.* **12** 1857–68

—— 1981 *Phys. Rev.* B **23** 6111–25

Fan C 1972a *Phys. Lett.* **39A** 136–6

—— 1972b *Phys. Rev.* B **6** 902–10

Knops H J F 1980 *Ann. Phys., NY* **128** 448–62

Nienhuis B 1984 *J. Stat. Phys.* **34** 731–61

Pearce P A 1985 *J. Phys. A: Math. Gen.* **18** 3217–26

—— 1987 *Phys. Rev. Lett.* **58** 1502–4

Pearce P A and Kim D 1987 *J. Phys. A: Math. Gen.* **20** 6471–85

Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106–9